Logical Foundations of Categorization Theory

ESSLLI 2021

Alessandra Palmigiano

Lecture 2

27 July 2021

Closure operators and complete lattices

If $(P, \leq)$ is a partially ordered set, a map $c : P \to P$ is a **closure operator** on $(P, \leq)$ if, for all x, y in P:

(a) $x \leq c(x)$;

(b) if $x \leq y$ then $c(x) \leq c(y)$;

(c) $c(c(x)) = c(x)$.

If c is a closure operator on $(P, \leq)$, let $\mathcal{C}_c := \{x \in P \mid x = c(x)\}$.
A **complete lattice** is a partially ordered set $(L, \leq)$ s.t:

(a) for every $X \subseteq L$ the greatest lower bound of X exists, i.e. an element $\bigwedge X$ s.t. $\bigwedge X \leq x$ for every $x \in X$ and for every $c \in L$, if $c \leq x$ for every $x \in X$ then $c \leq \bigwedge X$;

(b) for every $X \subseteq L$ the least upper bound of X exists, i.e. an element $\bigvee X$ s.t. $x \leq \bigvee X$ for every $x \in X$ and for every $c \in L$, if $x \leq c$ for every $x \in X$ then $\bigvee X \leq c$.

Closure operators/systems on complete lattices

Let $(L, \leq)$ be a complete lattice.

Fact 1

If $c : L \to L$ closure operator, and $X \subseteq \mathcal{C}_c := \{x \in P \mid x = c(x)\}$, then $\bigwedge X \in \mathcal{C}_c$.

Thus, $\mathcal{C}_c$ is a complete sub-meet-semilattice (aka a **closure system**) of L (but in general not its complete sublattice).

Fact 2

If $\mathcal{C} \subseteq L$ then the map $c_{\mathcal{C}} : L \to L$ defined by the assignment $a \mapsto \bigwedge\{x \in \mathcal{C} \mid a \leq x\}$ is a closure operator.

Fact 3

1. If $c : L \to L$ closure operator, then $c = c_{\mathcal{C}_c}$.
2. $\mathcal{C} \subseteq L$ closure system, then $\mathcal{C} = \mathcal{C}_{c_{\mathcal{C}}}$.

Hence, there is a perfect correspondence between closure operators and closure systems on complete lattices.

Closure operators and Galois connections

Let $(P, \leq)$ and $(Q, \preceq)$ be partial orders. A **Galois connection** is a pair of maps $\triangleright : P \to Q$ and $\blacktriangleright : Q \to P$ such that for every $x \in P$ and every $y \in Q$,

$$x \leq \blacktriangleright y \quad \text{iff} \quad y \preceq \triangleright x.$$

Fact 4

1. $x \leq \blacktriangleright\triangleright x$ and $y \leq \triangleright\blacktriangleright y$;
2. $x \leq x'$ implies $\triangleright x' \preceq \triangleright x$, and $y \preceq y'$ implies $\blacktriangleright y' \leq \blacktriangleright y$;
3. $\triangleright\blacktriangleright\triangleright x = \triangleright x$ and $\blacktriangleright\triangleright\blacktriangleright y = \blacktriangleright y$.

Thus, for every Galois connection, $\blacktriangleright\triangleright$ is a closure operator on $(P, \leq)$ and $\triangleright\blacktriangleright$ is a closure operator on $(Q, \preceq)$.

Polarities and Galois connections

A **polarity** or *formal context* is a triple $\mathbb{P} = (A, X, I)$ s.t. A and X sets and $I \subseteq A \times X$. Every polarity induces maps

$$(\cdot)^{\uparrow} : \mathcal{P}(A) \to \mathcal{P}(X) \quad \text{and} \quad (\cdot)^{\downarrow} : \mathcal{P}(X) \to \mathcal{P}(A),$$

$$B^{\uparrow} := \{x \in X \mid \forall a(a \in B \Rightarrow aIx)\}$$
$$Y^{\downarrow} := \{a \in A \mid \forall x(x \in Y \Rightarrow aIx)\}.$$

Fact 4

$(\forall B \subseteq A)(\forall Y \subseteq B)(B \subseteq Y^{\downarrow} \quad \text{iff} \quad Y \subseteq B^{\uparrow})$

Thus:

1. $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ form a Galois connection;
2. $(\cdot)^{\uparrow\downarrow} : \mathcal{P}(A) \to \mathcal{P}(A)$ and $(\cdot)^{\downarrow\uparrow} : \mathcal{P}(X) \to \mathcal{P}(X)$ are their associated closure operators.
3. The closure systems of $(\cdot)^{\uparrow\downarrow}$ and $(\cdot)^{\downarrow\uparrow}$ are dually isomorphic to each other via the restrictions of the maps $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$.

The concept lattice of a polarity

For every formal context $\mathbb{P} = (A, X, I)$, a **formal concept** of $\mathbb{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^{\uparrow} = Y$ and $Y^{\downarrow} = B$. The set B is the *extension* of c (also denoted $[\![c]\!]$), and Y is the *intension* of c (also denoted $(\![c]\!)$). Let $L(\mathbb{P})$ denote the set of the formal concepts of $\mathbb{P}$. Then the **concept lattice** of $\mathbb{P}$ is the complete lattice

$$\mathbb{P}^{+} := (L(\mathbb{P}), \bigwedge, \bigvee),$$

where for every $\mathcal{X} \subseteq L(\mathbb{P})$,

$$\bigwedge \mathcal{X} := (\bigcap_{c \in \mathcal{X}} [\![c]\!], (\bigcap_{c \in \mathcal{X}} [\![c]\!])^{\uparrow}) \quad \text{and} \quad \bigvee \mathcal{X} := ((\bigcap_{c \in \mathcal{X}} (\![c]\!))^{\downarrow}, \bigcap_{c \in \mathcal{X}} (\![c]\!)).$$

Then $\top^{\mathbb{P}^{+}} := \bigwedge \varnothing = (A, A^{\uparrow})$ and $\bot^{\mathbb{P}^{+}} := \bigvee \varnothing = (X^{\downarrow}, X)$, and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathbb{P})$,

$$c \leq d \quad \text{iff} \quad [\![c]\!] \subseteq [\![d]\!] \quad \text{iff} \quad (\![d]\!) \subseteq (\![c]\!).$$

Birkhoff's representation theorem of complete lattices

Theorem

Any complete lattice $\mathbb{L}$ is isomorphic to the concept lattice $\mathbb{P}^+$ of some formal context $\mathbb{P}$.

If $\mathbb{L} = (L, \leq)$ is a complete lattice, then let $\mathbb{P} := (L, L, \leq)$. We want to show that $\mathbb{L}$ is isomorphic to $\mathbb{P}^+$.

Preliminary observation

For every $a \in L$,

$$a^{\uparrow\downarrow} = \{b \in L \mid b \leq a\}.$$

Let us define $f : \mathbb{L} \to \mathbb{P}^+$ by $a \mapsto (a^{\uparrow\downarrow}, a^{\uparrow})$, and $g : \mathbb{P}^+ \to \mathbb{L}$ by $(\llbracket c \rrbracket, (\llbracket c \rrbracket)) \mapsto \bigvee \llbracket c \rrbracket$.
To show that f is an order-isomorphism (i.e. a surjective order-embedding) between $\mathbb{L}$ and $\mathbb{P}^+$, it is enough to show that

- f is surjective,
- f is an order-embedding.

Birkhoff's representation theorem of complete lattices

To show that f is surjective, let us show that for every $c = (\llbracket c \rrbracket, (\!|c|\!)) \in \mathbb{P}^+$,

$$c = f(g(c)).$$

For every $c = (\llbracket c \rrbracket, (\!|c|\!)) \in \mathbb{P}^+$,

$$f(g(c)) = f(\bigvee \llbracket c \rrbracket) = ((\bigvee \llbracket c \rrbracket)^{\uparrow\downarrow}, (\bigvee \llbracket c \rrbracket)^{\uparrow}),$$

hence to finish the proof that $f(g(c)) = c$, it is enough to show that $(\!|c|\!) = (\bigvee \llbracket c \rrbracket)^{\uparrow})$. By definition, $(\!|c|\!) = \llbracket c \rrbracket^{\uparrow}$, so we need to show that $\llbracket c \rrbracket^{\uparrow} = (\bigvee \llbracket c \rrbracket)^{\uparrow}$.

$$\begin{array}{rcll} \llbracket c \rrbracket^{\uparrow} & = & \{y \in L \mid \forall a (a \in \llbracket c \rrbracket \Rightarrow a \leq y)\} & \text{definition of } \llbracket c \rrbracket^{\uparrow} \\ & = & \{y \in L \mid \bigvee \llbracket c \rrbracket \leq y\} & \bigvee \text{ least upper bound} \\ & = & (\bigvee \llbracket c \rrbracket)^{\uparrow} & \text{definition of } (\bigvee \llbracket c \rrbracket)^{\uparrow} \end{array}$$

Birkhoff's representation theorem of complete lattices

Let us show that f is an order-embedding, i.e. that for every $a, b \in L$,

$$a \leq_{\mathbb{L}} b \quad \text{iff} \quad f(a) \leq_{\mathbb{P}^+} f(b).$$

By definition, $f(a) \leq_{\mathbb{P}^+} f(b)$ iff $[\![f(a)]\!] = a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow} = [\![f(b)]\!]$, so $a \leq_{\mathbb{L}} b$ implies that $a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow}$ since the composition of order-reversing maps is order preserving. For the converse implication, assume that $a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow}$; by the preliminary claim, this is equivalent to $\{x \in L \mid x \leq a\} \subseteq \{x \in L \mid x \leq b\}$; hence $a \in \{x \in L \mid x \leq b\}$, i.e. $a \leq b$, as required.