Closure operators and complete lattices

If \((P, \leq)\) is a partially ordered set, a map \(c : P \to P\) is a closure operator on \((P, \leq)\) if, for all \(x, y\) in \(P\):

(a) \(x \leq c(x)\);
(b) if \(x \leq y\) then \(c(x) \leq c(y)\);
(c) \(c(c(x)) = c(x)\).

If \(c\) is a closure operator on \((P, \leq)\), let \(C_c := \{x \in P \mid x = c(x)\}\).

A complete lattice is a partially ordered set \((L, \leq)\) s.t:

(a) for every \(X \subseteq L\) the greatest lower bound of \(X\) exists, i.e. an element \(\bigwedge X\) s.t. \(\bigwedge X \leq x\) for every \(x \in X\) and for every \(c \in L\), if \(c \leq x\) for every \(x \in X\) then \(c \leq \bigwedge X\);
(b) for every \(X \subseteq L\) the least upper bound of \(X\) exists, i.e. an element \(\bigvee X\) s.t. \(x \leq \bigvee X\) for every \(x \in X\) and for every \(c \in L\), if \(x \leq c\) for every \(x \in X\) then \(\bigvee X \leq c\).
Closure operators/systems on complete lattices

Let \((L, \leq)\) be a complete lattice.

**Fact 1**
If \(c : L \rightarrow L\) closure operator, and \(X \subseteq C_c := \{x \in P \mid x = c(x)\}\), then \(\bigwedge X \in C_c\).
Thus, \(C_c\) is a complete sub-meet-semilattice (aka a **closure system**) of \(L\) (but in general not its complete sublattice).

**Fact 2**
If \(C \subseteq L\) then the map \(c_C : L \rightarrow L\) defined by the assignment \(a \mapsto \bigwedge \{x \in C \mid a \leq x\}\) is a closure operator.

**Fact 3**

1. If \(c : L \rightarrow L\) closure operator, then \(c = c_{C_c}\).
2. \(C \subseteq L\) closure system, then \(C = C_{C_C}\).

Hence, there is a perfect correspondence between closure operators and closure systems on complete lattices.
Closure operators and Galois connections

Let \((P, \leq)\) and \((Q, \preceq)\) be partial orders. A **Galois connection** is a pair of maps \(\triangleright: P \to Q\) and \(\blacktriangleleft: Q \to P\) such that for every \(x \in P\) and every \(y \in Q\),

\[
x \leq \blacktriangleleft y \quad \text{iff} \quad y \preceq \triangleright x.
\]

**Fact 4**

1. \(x \leq \blacktriangleleft \blacktriangleleft x\) and \(y \preceq \triangleright \triangleright y\);
2. \(x \leq x'\) implies \(\triangleright x' \preceq \triangleright x\), and \(y \preceq y'\) implies \(\blacktriangleleft y' \leq \blacktriangleleft y\);
3. \(\triangleright \blacktriangleleft \triangleright x = \triangleright x\) and \(\blacktriangleleft \blacktriangleleft \blacktriangleleft y = \blacktriangleleft y\).

Thus, for every Galois connection, \(\triangleright \triangleright\) is a closure operator on \((P, \leq)\) and \(\blacktriangleleft \blacktriangleleft\) is a closure operator on \((Q, \preceq)\).
Polarities and Galois connections

A polarity or formal context is a triple \( \mathbb{P} = (A, X, I) \) s.t. \( A \) and \( X \) sets and \( I \subseteq A \times X \). Every polarity induces maps

\[
(\cdot)^\uparrow : \mathcal{P}(A) \to \mathcal{P}(X) \quad \text{and} \quad (\cdot)^\downarrow : \mathcal{P}(X) \to \mathcal{P}(A),
\]

\[
B^\uparrow := \{x \in X \mid \forall a (a \in B \Rightarrow alx)\}
\]

\[
Y^\downarrow := \{a \in A \mid \forall x (x \in Y \Rightarrow alx)\}.
\]

Fact 4

\((\forall B \subseteq A)(\forall Y \subseteq B)(B \subseteq Y^{\downarrow} \iff Y \subseteq B^{\uparrow})\)

Thus:

1. \((\cdot)^\uparrow\) and \((\cdot)^\downarrow\) form a Galois connection;
2. \((\cdot)^{\uparrow\downarrow} : \mathcal{P}(A) \to \mathcal{P}(A)\) and \((\cdot)^{\downarrow\uparrow} : \mathcal{P}(X) \to \mathcal{P}(X)\) are their associated closure operators.
3. The closure systems of \((\cdot)^{\uparrow\downarrow}\) and \((\cdot)^{\downarrow\uparrow}\) are dually isomorphic to each other via the restrictions of the maps \((\cdot)^\uparrow\) and \((\cdot)^\downarrow\).
The concept lattice of a polarity

For every formal context $\mathbb{P} = (A, X, I)$, a formal concept of $\mathbb{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^\uparrow = Y$ and $Y^\downarrow = B$. The set $B$ is the extension of $c$ (also denoted $\llbracket c \rrbracket$), and $Y$ is the intension of $c$ (also denoted $(\llbracket c \rrbracket)$). Let $L(\mathbb{P})$ denote the set of the formal concepts of $\mathbb{P}$. Then the concept lattice of $\mathbb{P}$ is the complete lattice

$$\mathbb{P}^+ := (L(\mathbb{P}), \land, \lor),$$

where for every $\mathcal{X} \subseteq L(\mathbb{P})$,

$$\land \mathcal{X} := \left( \bigcap_{c \in \mathcal{X}} \llbracket c \rrbracket, \bigcap_{c \in \mathcal{X}} (\llbracket c \rrbracket)^\uparrow \right) \quad \text{and} \quad \lor \mathcal{X} := \left( (\bigcap_{c \in \mathcal{X}} \llbracket c \rrbracket)^\downarrow, \bigcap_{c \in \mathcal{X}} \llbracket c \rrbracket \right).$$

Then $\top^{\mathbb{P}^+} := \land \emptyset = (A, A^\uparrow)$ and $\bot^{\mathbb{P}^+} := \lor \emptyset = (X^\downarrow, X)$, and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathbb{P})$,

$$c \leq d \quad \text{iff} \quad \llbracket c \rrbracket \subseteq \llbracket d \rrbracket \quad \text{iff} \quad (\llbracket d \rrbracket) \subseteq (\llbracket c \rrbracket).$$
Birkhoff’s representation theorem of complete lattices

Theorem

Any complete lattice $\mathbb{L}$ is isomorphic to the concept lattice $\mathbb{P}^+$ of some formal context $\mathbb{P}$.

If $\mathbb{L} = (L, \leq)$ is a complete lattice, then let $\mathbb{P} := (L, L, \leq)$. We want to show that $\mathbb{L}$ is isomorphic to $\mathbb{P}^+$.

Preliminary observation

For every $a \in L$,

$$a^{\uparrow \downarrow} = \{ b \in L \mid b \leq a \}.$$ 

Let us define $f : \mathbb{L} \to \mathbb{P}^+$ by $a \mapsto (a^{\uparrow \downarrow}, a^\uparrow)$, and $g : \mathbb{P}^+ \to \mathbb{L}$ by $(\llbracket c \rrbracket, (\llbracket c \rrbracket)) \mapsto \bigvee \llbracket c \rrbracket$.

To show that $f$ is an order-isomorphism (i.e. a surjective order-embedding) between $\mathbb{L}$ and $\mathbb{P}^+$, it is enough to show that

- $f$ is surjective,
- $f$ is an order-embedding.
Birkhoff’s representation theorem of complete lattices

To show that $f$ is surjective, let us show that for every 
$c = ([c], (c)) \in \mathbb{P}^+$,

$c = f(g(c))$.

For every $c = ([c], (c)) \in \mathbb{P}^+$,

$$f(g(c)) = f(\bigvee [c]) = ((\bigvee [c])^{\uparrow \downarrow}, (\bigvee [c])^{\uparrow})$$

hence to finish the proof that $f(g(c)) = c$, it is enough to show 
that $([c]) = (\bigvee [c])^{\uparrow})$. By definition, $([c]) = [c]^{\uparrow}$, so we need to 
show that $[c]^{\uparrow} = (\bigvee [c])^{\uparrow}$.

$$
\begin{align*}
[c]^{\uparrow} & = \{ y \in L \mid \forall a (a \in [c] \Rightarrow a \leq y) \} & \text{definition of } [c]^{\uparrow} \\
& = \{ y \in L \mid \bigvee [c] \leq y \} & \text{ } \bigvee \text{ least upper bound} \\
& = (\bigvee [c])^{\uparrow} & \text{definition of } (\bigvee [c])^{\uparrow}
\end{align*}
$$
Birkhoff’s representation theorem of complete lattices

Let us show that $f$ is an order-embedding, i.e. that for every $a, b \in L$,

$$a \leq_L b \iff f(a) \leq_{\mathbb{P}^+} f(b).$$

By definition, $f(a) \leq_{\mathbb{P}^+} f(b)$ iff $\llbracket f(a) \rrbracket = a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow} = \llbracket f(b) \rrbracket$, so $a \leq_L b$ implies that $a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow}$ since the composition of order-reversing maps is order preserving. For the converse implication, assume that $a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow}$; by the preliminary claim, this is equivalent to $\{x \in L \mid x \leq a\} \subseteq \{x \in L \mid x \leq b\}$; hence $a \in \{x \in L \mid x \leq b\}$, i.e. $a \leq b$, as required.