Logical Foundations of Categorization Theory

ESSLLI 2021

Alessandra Palmigiano

Lecture 1

26 July 2021

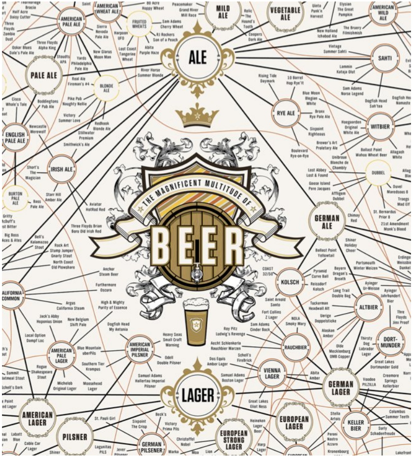
Categorization theory

From Wikipedia:

Categorization is the process in which ideas and objects are recognized, differentiated, and understood.

Ideally, a category illuminates a relationship between the subjects and objects of knowledge.

Categorization is fundamental in language, prediction, inference, and decision-making.



Mainstream views on categorization in empirical sciences

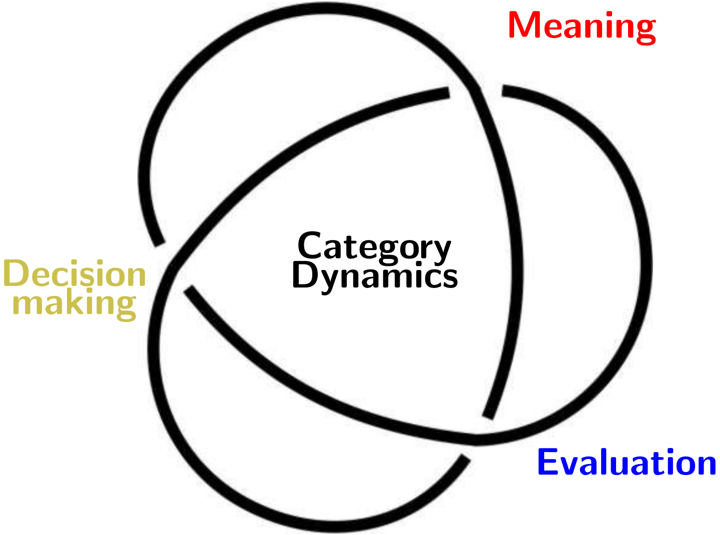
- categories are both **fuzzy** and **internally coherent**;
- categories can be described in terms of **objects** and **features**;
- categories do not occur in isolation but within **categorization systems**;
- categories arise on the basis of and allow to make **similarity judgments**, which underlie **decision-making processes**;
- **partial membership** in a category is the usual situation;
- the whole list of the defining features of a category is almost never specified in its entirety; some features are **taken for granted**, they are assumed **by default**;
- within categorization systems, categories **are created and evolve** by the interrelated actions of many actors;
- the **persistence** of categories depends on their being used or accepted by more than one actor.

3

What we have learned so far (as formal philosophers)

- Categories are the most basic **cognitive tools**.
- Categories are the building blocks of **meaning**.
- Categories are specified both **extensionally** and **intensionally**.
- Categories do not occur in isolation, but in **hierarchies**.
- Categories **mediate social interaction**.
- Categories provide the background for **evaluation**.
- Categories underlie all **decision-making** processes.
- Categorization theory is a **powerful unifying tool** to systematically connect phenomena cropping up and studied in different disciplines within **one and the same theoretical framework**.

The research program in a picture



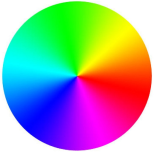
The classical view

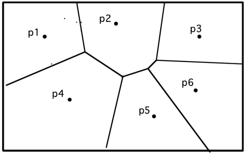
- Categorization is a **deductive** process of verifying whether a certain object satisfies each feature defining a given category.
- Categories resulting from this process have **sharp boundaries** (membership is either 0% or 100%),
- categories are represented **equally well by any** of their **members**.

Difficulties

- accommodate a **new objects** or entity which would intuitively be part of a given category but does not share all the defining features of the category.
- providing an **exhaustive list** of defining features (Wittgenstein: what is a game?);
- dealing with **unclear cases** (is it blue or is it green?);
- the existence of members which are **better representatives** of the whole class than others.

The prototype view





- categorization is the **inductive process** of finding the best match between the features of an object and those of the closest prototype(s);
- membership in a category is **not decided** through the satisfaction of an **exhaustive list of features**.
- allows for **unclear cases**;
- agrees with intuition that membership in most categories is a **matter of degrees**,
- certain members are **more central** (or prototypical) to a category than others (robins are prototypical birds, penguins are not.)

7

The exemplar view

- **How do we generate prototypes** in our minds? Prototype theory does not have an answer to this issue.
- the exemplar view: individuals make category judgments by comparing new stimuli with instances already stored in memory (the "exemplars").

Difficulties: circularity

- the existence of instances or prototypes of a given category presupposes that this category has already been defined.
- similarity-based views (e.g. exemplar/prototype) fail to explain 'why we have the categories we have', i.e. why certain categories seem to be more cogent and coherent than others.
- similarity might be **imposed rather than discovered** (do things belong in the same category because they are similar, or are they similar because they belong in the same category?), i.e. similarity might be the **effect of conceptual coherence** rather than its cause.

8

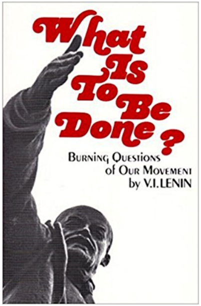
The theory-based view

- it tries to address ‘why we have the categories we have’, i.e. why certain categories are more cogent/coherent than others.
- theory-based view: categories arise in connection with **theories** (broadly understood so as to include also informal explanations).
- Coherence of categories proceeds from the coherence of their associated theories.
- It accounts for the formation of categories of **disparate entities**;
- it accounts for **goal-based categories** (what makes a good birthday present for an 18-yr old girl);

Difficulties: circularity

- categories themselves underlie theory-formation!
- how do changes in the theories account for changes in the categories?

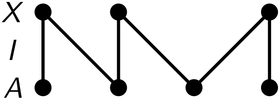
What Is to Be Done?



- a **formal model-based approach**:
- try and formally account for as many insights as possible into one coherent framework
- the framework will provide information on which desiderata can coexist, and how.

Formal contexts [Birkhoff 1940, Wille 1971]

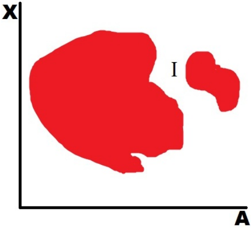
Formal contexts (A, X, I) are abstract representations of databases:



A: set of *Objects*

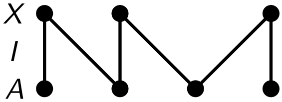
X: set of *Features*

$I \subseteq A \times X$. Intuitively, aIx reads: object a has feature x



Formal contexts [Birkhoff 1940, Wille 1971]

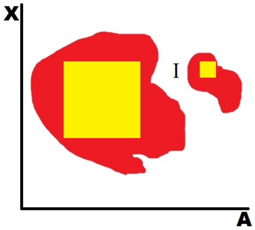
Formal contexts (A, X, I) are abstract representations of databases:



A: set of *Objects*

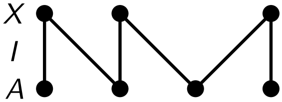
X: set of *Features*

$I \subseteq A \times X$. Intuitively, aIx reads: object a has feature x



Formal contexts [Birkhoff 1940, Wille 1971]

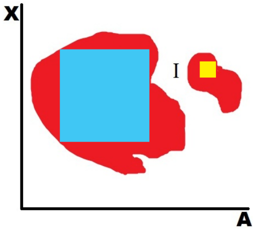
Formal contexts (A, X, I) are abstract representations of databases:



A: set of *Objects*

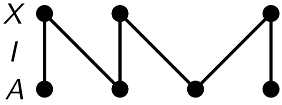
X: set of *Features*

$I \subseteq A \times X$. Intuitively, aIx reads: object a has feature x



Formal contexts [Birkhoff 1940, Wille 1971]

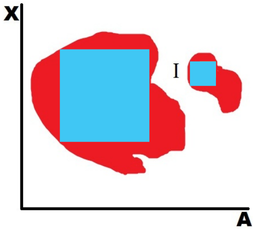
Formal contexts (A, X, I) are abstract representations of databases:



A: set of *Objects*

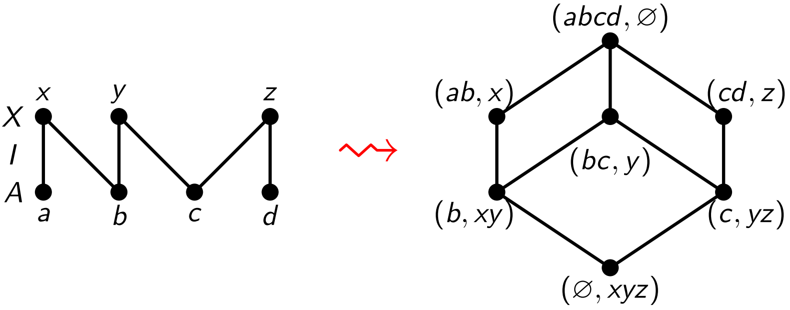
X: set of *Features*

$I \subseteq A \times X$. Intuitively, aIx reads: object a has feature x



Formal concepts: "rectangles" maximally contained in I

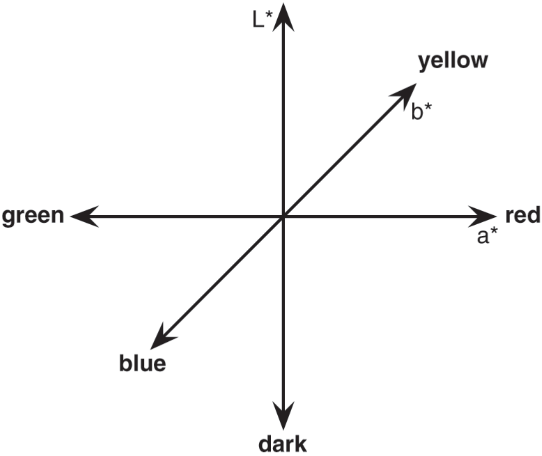
Concept lattices



Formal concepts can be ordered and form a complete lattice (in general non-distributive).

Every complete lattice is isomorphic to the concept lattice of some formal context.

Conceptual spaces [Gärdenfors 2004]

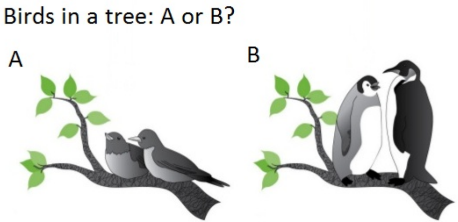


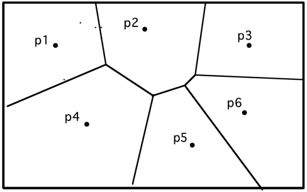
Objects (= points in the space) uniquely identified by their scores (= value of coordinate) along each parameter (= dimension).

Concepts are **convex subsets** of the space.

13

Core concept: Typicality





- in conceptual spaces, the **prototype** of a formal concept is defined as the **geometric center of that concept**;
- the closer (i.e. more similar) an object is to the prototype, the stronger its typicality.
- Advantage: **visually appealing**;
- Disadvantage: does not explain the **role of agents** in establishing the typicality of an object relative to a category.

Are categories sets?

Yes. However, **not all sets are categories**:

- there is no such thing as the category of **non-apples**.
- Aristotle:

  > *Forms of speech are either simple or composite. Examples of the latter are such expressions as 'the man runs', 'the man wins'; of the former 'man', 'ox', 'runs', 'wins'.*

  simple forms of speech are **categories**. Hence, **truth values do not apply to categories.**

Restructuring Logic: three quotes from R. Wille

"For a mathematical theory of concepts and concept hierarchies, we obviously need a mathematical model that allows to **speak** *mathematically about objects, attributes, and relationships which indicate that an object has an attribute."*

"The connections of logic to reality have been **narrowed** *since* **Frege's turn** *to predicate logic, the leading paradigm of mathematical logic today. Thus, restructuring has to establish a* **broader** *understanding of mathematical logic, in particular, by elaborating the* **pragmatic dimension***."*

"a one-sided priority [...] of formal logic leads to view concepts through the conventional perspective and to disregard the primarily **personal nature of concepts***."*

Taking stock 1/3

- categories are the most basic cognitive tools, and are key to **meaning, evaluation and decision-making**.
- categories shape and are shaped by **social interaction**.
- **various extant theories/views** on categorization, each capturing important aspects of what categories are and do, but none “with all the answers”.
- some views seem to **clash with each other**: sharp vs fuzzy? any representative vs prototypes? uniform vs internally graded? similarity vs internal coherence?
  - can we reconcile some of these (seeming) dichotomies?
- various proposed formal models for categories/concepts; best known: Gärdenfors' **conceptual spaces** and Wille's **formal contexts**.
  - how do these models relate with each other?

Taking stock 2/3

Our proposal

- Wille's formal contexts as models for **basic propositional lattice logic**;
- basic propositional lattice logic as the **basic propositional logic of categories and concepts**;
- basic normal (lattice-based) modal logic as **epistemic logic of categories and concepts**;
- this framework explicitly accommodates both the subjective and intersubjective perspectives in categorization;
- epistemic interpretation carries over to the meaning of well known **epistemic axioms**;
- **typicality** and **default** captured by well known tools of epistemic logic (common knowledge, transitivity)

Taking stock 3/3

Advancements of our proposal

- our framework systematically connects and embeds categorization theory into the formal theory of **agency**;
- insights from **classical, prototype** and **theory-based** views brought coherently together in one single formal framework:
  - extensional and intensional views perfectly balanced (as required by **classical** theory);
  - internal coherence (as required by **theory-based** view) mathematically captured as Galois-stability;
  - **typicality** (as required by **prototype** view) formalized explicitly in terms of the intersubjective component;
  - resulting categories are both **sharp** and **internally graded**;
  - **similarity within each category** can be also defined in terms of intersubjectivity.

Unification via closure operators 1/2

Closure operator

$(P, \leq)$ poset, $c : P \to P$ s.t. for all x, y in P:

- $x \leq c(x)$;
- if $x \leq y$ then $c(x) \leq c(y)$;
- $c(c(x)) = c(x)$.

Examples

- X topological space. Then $Y \mapsto \bigcap\{C \in \mathit{Closed}(X) \mid Y \subseteq C\}$ defines a closure operator on $\mathcal{P}(X)$.
- L logic. Then $\Gamma \mapsto \{\phi \mid \Gamma \vdash \phi\}$ defines a cl. op. on $\mathcal{P}(\mathbf{Fm})$.
- X Euclidean space. Then $Y \mapsto \bigcap\{C \in \mathit{Convex}(X) \mid Y \subseteq C\}$ defines a closure operator on $\mathcal{P}(X)$.
- V vector space. Then $Y \mapsto \mathit{Subspace}(Y)$ defines a closure operator on $\mathcal{P}(V)$.

Unification via closure operators 2/2

Closure operators and lattices

- $c : P \to P$ completely determined by $Cl(c) = \{x \in P \mid x = c(x)\}$ its **closed elements**.
- For any $c : P \to P$, $(Cl(c), \bigcap, \bigvee)$ is a **complete lattice**.
- Conversely, if $\mathbb{L}$ complete lattice then $\mathbb{L} \cong Cl(c)$ for some $c : \mathcal{P}(X) \to \mathcal{P}(X)$ for some set X.
- (A, X, I) formal context. The concept lattice of (A, X, I) is isomorphic to the lattice of closed sets of the closure operator $c : P(A) \to P(A)$ defined by mapping any subset $B \subseteq A$ to the extension of the concept generated by B.

Hence, via Birkhoff's representation theorem, formal contexts provide the most general environment to describe and reason about categorical hierarchies, into which all other models can be embedded.

Conclusions

- we have laid the **groundwork** for the logical foundations of categorization theory;
- we have used **established methodologies** from the mathematical theory of modal logic, and used them to give a novel interpretation/connection with logic to well known structures.
- categories have been **at the heart of logic** since Aristotle, but much less so since Frege's turn in mathematical logic;
- we can use the tools of mathematical logic to restore the centrality of categories in logic, and their key role within a formal theory of **agency**.