1 Order-theoretic notions

**Definition 1** (Maximal and minimal elements). Let \((P, \leq)\) be a partially ordered set and let \(Q \subseteq P\). Then:

(a) \(a \in Q\) is a maximal element of \(Q\) if, for all \(x \in P\), if \(a \leq x\) and \(x \in Q\) then \(a = x\);

(b) \(a \in Q\) is a minimal element of \(Q\) if, for all \(x \in P\), if \(x \leq a\) and \(x \in Q\) then \(a = x\);

(c) \(a \in Q\) is the greatest (or maximum) element of \(Q\) if \(x \leq a\) for all \(x \in Q\);

(d) \(a \in Q\) is the least (or minimum) element of \(Q\) if \(a \leq x\) for all \(x \in Q\).

We denote the set of maximal elements of \(Q\) by \(\text{Max}Q\) and the set of minimal elements of \(Q\) by \(\text{Min}Q\).

**Exercise 1.** Prove that if \(y \in Q\) is the maximum element of \(Q\), then \(\text{Max}Q = \{y\}\); dually, if \(y \in Q\) the minimum element of \(Q\), then \(\text{Min}Q = \{y\}\). Deduce that if the maximum (or minimum) element of \(Q\) exists, then it is unique.

**Definition 2** (Upper bound and lower bound). Let \((P, \leq)\) be a partially ordered set and let \(Q \subseteq P\). Then:

(a) \(a \in P\) is an upper bound of \(Q\) if \(x \leq a\) for all \(x \in Q\);

(b) \(a \in P\) is a lower bound of \(Q\) if \(a \leq x\) for all \(x \in Q\);

The set of all upper bounds of \(Q\) is denoted by \(Q^u\) and the set of all lower bounds by \(Q^\ell\). If the minimum element (denoted by \(\vee Q\)) of \(Q^u\) exists, then \(\vee Q\) is called the least upper bound (or supremum) of \(Q\). Dually, if the maximum element (denoted by \(\wedge Q\)) of \(Q^\ell\) exists, then \(\wedge Q\) is called the greatest lower bound (or infimum) of \(Q\).

**Exercise 2.** Give an example of a partially ordered set \((P, \leq)\) and \(Q \subseteq P\) in which \(\vee Q \notin Q\) and an example of a partially ordered set \((P, \leq)\) and \(Q \subseteq P\) in which \(\wedge Q \notin Q\).
2  Lattices and closure systems

Definition 3 (Bounded lattice). A lattice is a partially ordered set \((L, \leq)\) such that:

(a) the infimum \(a \wedge b\) of the set \(\{a, b\}\) exists for any \(a, b \in L\);

(b) the supremum \(a \vee b\) of the set \(\{a, b\}\) exists for any \(a, b \in L\);

(c) the maximum \(\top\) of \(L\) exists;

(d) the minimum \(\bot\) of \(L\) exists.

Such a lattice is complete if

(a') \(\bigwedge X \in L\) exists for every \(X \subseteq L\);

(b') \(\bigvee X \in L\) exists for every \(X \subseteq L\).

A partial order for which (a) and (c) hold is a meet-semilattice; a partial order for which (b) and (d) hold is a join-semilattice. A partial order for which (a') hold is a complete meet-semilattice; a partial order for which (b') holds is a complete join-semilattice.

Exercise 3. For any partial order \((P, \leq)\),

1. Prove that if \(X \subseteq Y \subseteq P\) then \(\bigvee Y \leq \bigvee X\) and \(\bigwedge Y \leq \bigwedge X\) whenever they exist;

2. Deduce that if \(\top\) exists then \(\top = \bigwedge \emptyset\) and if \(\bot\) exists then \(\bot = \bigvee \emptyset\).

Exercise 4. Prove that if \((L, \leq)\) is a complete meet-semilattice, then \((L, \leq)\) is a complete lattice. Hint: Let \(X \subseteq L\); to show that \(\bigvee X\) exists, consider \(X^\complement\) (cf. Definition ??).

Definition 4 (Closure system). If \((L, \leq)\) is a complete lattice, a closure system of \((L, \leq)\) is a subset \(C \subseteq L\) such that, for every \(X \subseteq L\), if \(X \subseteq C\) then \(\bigwedge X \in C\).

Exercise 5. Let \((L, \leq)\) be a complete lattice. Prove that \(\top \in C\) for any closure system of \((L, \leq)\).

Exercise 6. For any topological space \((X, \tau)\), show that the set \(C_\tau := \{A^\complement \mid A \in \tau\}\) is a closure system, where \(A^\complement\) is the relative complement of \(A\) with respect to \(X\).

Exercise 7. For any logical system \(\mathcal{S}\), let the map \(c_\vdash : \mathcal{P}(\text{Fm}) \to \mathcal{P}(\text{Fm})\) be defined by the assignment \(\Gamma \mapsto \{\varphi \mid \Gamma \vdash \varphi\}\). Show that \(C_{\vdash} = \{\Gamma \subseteq \text{Fm} \mid \forall \varphi(\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma)\}\) is a closure system. Which set of formulas is \(\bot_{c_\vdash}\)?

Exercise 8. Let \((L, \leq)\) be a complete lattice. Prove that, if \(C \subseteq L\) is a closure system of \((L, \leq)\), then for every \(Y \subseteq C\), \(\bigvee_C Y = c_\vdash(\bigvee Y)\), where the map \(c_\vdash : L \to L\) is defined by the assignment \(a \mapsto \bigwedge\{x \in C \mid a \leq x\}\). (Notice that: \(\bigvee_C Y\) is the supremum of \(Y\) in \(C\) (cf. Exercise 5) and \(\bigvee Y\) is the supremum of \(Y\) in \(L\).)
3 Birkhoff’s representation theorem for complete lattices

Definition 5 (Homomorphism). Let $L$ and $K$ be bounded lattices. A map $f : L \to K$ is said to be a homomorphism (or bounded lattice homomorphism) if

(a) for all $a, b \in L$, $f(a \land L b) = f(a) \land K f(b)$;

(b) for all $a, b \in L$, $f(a \lor L b) = f(a) \lor K f(b)$;

(c) $f(\bot L) = \bot K$;

(d) $f(\top L) = \top K$.

If $L$ and $K$ are complete lattices, $f : L \to K$ is said to be a complete homomorphism if

(a') for any set $X \subseteq L$, $f(\bigwedge L X) = \bigwedge K \{ f(a) \mid a \in L \}$;

(b') for any set $X \subseteq L$, $f(\bigvee L X) = \bigvee K \{ f(a) \mid a \in L \}$.

$f$ is said to be an isomorphism (or lattice isomorphism) if $f$ is a bijection.

Exercise 9. Let $L$ and $K$ be bounded lattices. Show that $f : L \to K$ is an isomorphism if and only if it is surjective and an order-embedding.  \footnote{Let $(P, \subseteq)$ and $(Q, \subseteq)$ be partially ordered sets. A map $f : P \to Q$ is an order-embedding if, for all $x, y \in P$, $x \leq y$ in $P$ if and only if $f(x) \leq f(y)$ in $Q$.}

Definition 6. For every formal context $\mathbb{P} = (A, X, I)$, a formal concept of $\mathbb{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^\uparrow = Y$ and $Y^\uparrow = B$ (cf. see Definition Exercise/Example 4 in Lecture 2). The set $B$ is the extension of $c$, which we will sometimes denote $[c]$, and $Y$ is the intension of $c$, sometimes denoted $(c]$. Let $L(\mathbb{P})$ denote the set of the formal concepts of $\mathbb{P}$. Then the concept lattice of $\mathbb{P}$ is the complete lattice $\mathbb{P}^+ := (L(\mathbb{P}), \bigwedge, \bigvee)$, where for every $\mathcal{X} \subseteq L(\mathbb{P})$,

$\bigwedge \mathcal{X} := (\bigcap_{c \in \mathcal{X}} [c], (\bigcap_{c \in \mathcal{X}} ([c])^{\uparrow})$ and $\bigvee \mathcal{X} := ((\bigcap_{c \in \mathcal{X}} ([c])^{\uparrow}, \bigcup_{c \in \mathcal{X}} ([c])^{\uparrow})$.

and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathbb{P})$,

$c \leq d$ iff $[c] \subseteq [d]$ iff $([d]) \subseteq ([c])$.

Exercise 10. Prove that $\top^{\mathbb{P}^+} := \bigwedge \mathcal{X} = (A, A^{\uparrow})$ and $\bot^{\mathbb{P}^+} := \bigvee \mathcal{X} = (X^{\downarrow}, X)$.

Exercise 11. Compute the concept lattices associated with the polarities $\mathbb{P} = (A, X, I)$, such that $A = \{a, b, c\}$, $X = \{x, y, z\}$ and $I \subseteq A \times X$ in the following cases:

(1) $I = \{(a, x), (b, y), (c, z)\}$;

(2) $I = \{(a, x), (b, x), (b, y), (c, y), (c, z)\}$;

(3) $I = \{(a, x), (a, y), (b, x), (b, z), (c, y), (c, z)\}$.

Exercise 12 (Birkhoff’s representation theorem). Complete the proof of Birkhoff’s theorem in Lecture 2.